4 May 2023
Thursday.

Assignment – II

Q1: (i) $e^{-3t} \dfrac{\sin t}{t}$

Solⁿ (i) $L\{\sin t\} = \dfrac{1}{p^2+1}$

$$L\left\{\frac{\sin t}{t}\right\} = \int_p^{\infty} \frac{1}{p^2+1}\, dp \Rightarrow \left[\tan^{-1}\right]_p^{\infty}$$

$$= \cot^{-1} p. = f(p).$$

$$L\{e^{-3t} f(p)\} = \cot^{-1}(p+3)$$

(ii) $f(t) = \int_0^t \int_0^t \sin u\, du\, du.$

(ii) $L\{\sin u\} = \dfrac{1}{p^2+1} = f(p)$

$$L\left\{\int_0^t \int_0^t \sin u\, du\, du\right\} = \frac{1}{p^2} f(p)$$

$$= \frac{1}{p^2}\left(\frac{1}{p^2+1}\right).$$

Q2: (ii) $f(t) = \begin{cases} 0, & 0 < t < 2 \\ -1, & 2 < t < 4. \end{cases}$

Solⁿ: (ii)

$$\frac{1}{1-e^{-pT}} \int_0^T e^{-pt} f(t)\, dt.$$

$$= \frac{1}{1-e^{-4p}}\left[\int_0^2 e^{-pt}\, dt + \int_2^4 e^{-pt}(-1)\, dt\right].$$

$$= \frac{1}{1-e^{-4p}}\left[\int_0^2 e^{-pt}dt - \int_2^4 e^{-pt}dt\right].$$

$$= \frac{1}{1-e^{-4p}}\left[\left(\frac{-e^{-pt}}{p}\right)_0^2 - \left(\frac{-e^{-pt}}{p}\right)_2^4\right].$$

$$= \frac{1}{1-e^{-4p}}\left[\frac{1}{p} - \frac{e^{-2p}}{p} - \frac{e^{-4p}}{p} + \frac{e^{-2p}}{p}\right].$$

$$= \frac{1}{p(1-e^{-4p})}\left[1 - e^{-2p} - e^{-4p} + e^{-2p}\right].$$

$$= \frac{1}{p(1-e^{-4p})}\left[(1-e^{-4p}) + 2e^{-2p}\right].$$

$$= \frac{1}{p} + \frac{2e^{-2p}}{p(1-e^{-4p})}$$

$$= \frac{1}{p} + \frac{2}{p(e^{2p}-e^{-2p})}$$

Q3 (i) $L^{-1}\left\{\frac{5p+3}{(p-1)(p^2+2p+5)}\right\}.$

Solⁿ (i) $\frac{5p+3}{(p-1)(p^2+2p+5)} = \frac{A}{p-1} + \frac{Bp+C}{p^2+2p+5}$

$$\frac{5p+3}{\cancel{(p-1)(p^2+2p+5)}} = \frac{A(p^2+2p+5) + Bp+C(p-1)}{\cancel{(p-1)(p^2+2p+5)}}$$

$5p+3 = p^2(A+B) + p(2A-B+C) + 5A - C.$

$A+B = 0$

$2A - B + C = 05$ $\Rightarrow$ $\boxed{A=1}$ $\boxed{B=-1}$

$5A - C = 3.$ $\boxed{C=2}$

$$= \frac{1}{p-1} - \frac{p}{p^2+2p+5} + \frac{2}{p^2+2p+5}$$

$$= L^{-1}\left\{ \frac{1}{p-1} - \frac{p}{(p+1)^2+2^2} + \frac{2}{(p+1)^2+2^2} \right\}.$$

$$= e^{t} - \cos 2t + \sin 2t.$$

Q3 (ii) Evaluate $L\{e^{-t}[t-u(t-2)]\}$.

Soln: $L\{e^{-t}\cdot t - e^{-t}u(t-2)\}$.

$$= L\{e^{-t}\cdot t\} - L\{e^{-t}u(t-2)\}.$$

$$= \frac{1}{(p+1)^2} - L\{e^{-t}\} \int_0^\infty e^{-pt}u(t-2)\,dt$$

$$= \frac{1}{(p+1)^2} - \frac{e^{-2(p+1)}}{p+1}.$$

Q.4 (i) Using convolution theorem solve

$$L^{-1}\left\{ \frac{1}{(p^2+a^2)^2} \right\}.$$

Soln: Let $F(p) = \frac{1}{(p^2+a^2)^2} \Rightarrow F(t) = \frac{\sin at}{a}.$

$$G(p) = \frac{1}{(p^2+a^2)^2} \Rightarrow G(t) = \frac{\sin at}{a}.$$

$$\Rightarrow G(t-u) = \frac{\sin a(t-u)}{a}.$$

According to convolution theorem:

$$L^{-1}\{f(p)g(p)\} = \int_0^t F(u)\,G(t-u)\,du$$

$$= \frac{1}{a^2}\int_0^2 \sin au \cdot \sin a(t-u)\,du.$$

$$= \frac{1}{a^2}\int_0^2 \frac{1}{2}(\cos at + \cos(2u-t)\,du$$

$$= \frac{1}{2a^2}\int_0^t \cos a(2u-t) + \cos at\;du.$$

$$= \frac{1}{2a^2}\left[\frac{\sin a(2u-t)}{a}\right]_0^t + \cos t.$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{a} + \cos at\right].$$

$$= \frac{\sin at}{2a^3} + \frac{\cos at}{2a^2}.$$